Seat No. : _______________

JB-111

January-2018

M.Sc., Sem.-I

404 : Mathematics
(Ordinary Differential Equations)
(Old)

Time : 3 Hours] **[Max. Marks : 70**

1. (a) Find the general solution of the equation $(1 + x^2)\, y'' + xy' + y = 0$ in terms of power series in x. **7**

OR

Find the solutions of the following initial value problems :

(i) $y'' + 4y' + 5y = 0$, $y(0) = 1$, $y'(0) = 0$.

(ii) $y'' - 5y' + 6y = 0$, $y(1) = e^2$, $y'(1) = 3e^2$.

(b) Attempt any **two** : **4**

(i) Show that the series $y = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$.. satisfies $y'' = -y$.

(ii) Solve : $(x \log x)\, y' + y = 3x^3$.

(iii) Solve $y' = y$ in terms of a power series in x.

(c) Answer very briefly : **3**

(i) What is the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$?

(ii) What is the general solution of $y'' + 2y' - 8y = 0$?

(iii) Is the point $x = 0$ an ordinary point of the equation $x^2y'' + xy' - y = 0$?

2. (a) Find two independent Frobenius series solutions of any **one** of the following equations : **7**

(i) $4xy'' + 2y' + y = 0$

(ii) $2x^2y'' + x(2x + 1)\, y' - y = 0$

(b) Attempt any **two** : **4**

(i) Show that $e^x = \lim\limits_{b\to\infty} F\left(a, b, a, \frac{x}{b}\right)$.

(ii) Show that $F'(a, b, c, x) = \frac{ab}{c} F\,(a + 1, b + 1, c + 1, x)$.

(iii) Determine the nature of the point $x = \infty$ for the Bessel's equation

$x^2y'' + xy' + (x^2 - p^2)\, y = 0$.

(c) Answer very briefly : **3**

(i) Write down the Gauss's hypergeometric equation.

(ii) Is $x = -1$ a regular singular point of the equation $(x^2 - 1)\, y'' + xy' + y = 0$?

(iii) Write down the general solution of the Gauss's hypergeometric equation near the singular point $x = 0$.

3. (a) Attempt any **one** : **7**

(i) Derive the recursion formula for the Legendre polynomials and calculate $P_2(x)$, $P_3(x)$, $P_4(x)$ by taking $P_0(x) = 1$ and $P_1(x) = x$.

(ii) Prove that $\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = 0$, if $m \neq n$.

(b) Attempt any **two** : **4**

(i) Show that $T_n(x) + T_{n-2}(x) = 2xT_{n-1}(x)$.

(ii) Show that $T_n(x) = \frac{1}{2}\left[\left(x + \sqrt{x^2 - 1}\right)^n + \left(x - \sqrt{x^2 - 1}\right)^n\right]$.

(iii) Find the first two terms of the Legendre series of $f(x) = e^x$.

(c) Answer very briefly : **3**

(i) What is the value of $\int_{-1}^{1} P_5(x)^2\, dx$?

(ii) Write down the Rodrigues' formula for the Legendre polynomials.

(iii) Write down the generating function of the Legendre polynomials.

4. (a) Attempt any **one** : **7**

(i) Prove that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\, \sin x$ and $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\, \cos x$.

(ii) Prove that $\int_{0}^{1} xJ_p(\lambda_m x)\, J_p(\lambda_n x)\, dx = 0$, if $m \neq n$.

(b) Attempt any **two** : **4**

(i) Show that $2J_p'(x) = J_{p-1}(x) - J_{p+1}(x)$

(ii) Show that $\frac{2p}{x} J_p(x) = J_{p-1}(x) + J_{p+1}(x)$

(iii) Show that $\cos x = J_0(x) - 2J_2(x) + 2J_4(x) - \ldots$

(c) Answer very briefly : **3**

(i) Show that between any two positive zeros of $J_0(x)$ there is a zero of $J_1(x)$.

(ii) Write down the Bessel's integral formula.

(iii) Express $J_2(x)$ in terms of $J_0(x)$ and $J_1(x)$.

5. (a) Attempt any **one** : **7**

(i) Find the exact solution of the initial value problem

$y' = y^2$, $y(0) = 1$.

Find successive approximations $y_0(x)$, $y_1(x)$, $y_2(x)$, $y_3(x)$ using Picard's method.

(ii) Solve the following initial value problem by Picard's method.

$$\begin{cases} \dfrac{dy}{dx} = z, \ \ y(0) = 1, \\ \dfrac{dz}{dx} = -y, \ \ z(0) = 0. \end{cases}$$

(b) Attempt any **two** : **4**

(i) State (only) Picard's theorem.

(ii) Show that $f(x, y) = y^{1/2}$ does not satisfy a Lipschitz condition on the rectangle $|x| \leq 1$ and $0 \leq y \leq 1$.

(iii) Does $f(x, y) = y^{1/2}$ satisfy a Lipschitz condition on the rectangle $|x| \leq 1$ and $c \leq y \leq d$, where $0 < c < d$? Justify your answer.

(c) Answer very briefly : **3**

(i) State Lipschitz condition.

(ii) State Peano's theorem.

(iii) Why Picard's theorem is known as local existence and uniqueness theorem ?

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1. (a) Attempt any **one** :

   (i) Find the general solution of the equation $y'' + xy' + (3x + 2)y = 0$ near $x = 0$. **7**

   (ii) Find the solution $y(x)$ of the equation $y'' + y' - xy = 0$, such that $y(0) = 0$ and $y'(0) = 1$.

   (b) Attempt any **one** : **7**

   (i) Express the function $\sin^{-1} x$ in the form of a power series about $x = 0$.

   (ii) Express the function $(1 + x)^p$ in the form of a power series about $x = 0$.

2. (a) Attempt any **one** : **7**

   (i) Solve the equation $4xy'' + 2y' + y = 0$ near $x = 0$.

   (ii) Solve the equation $2xy'' + (3 - x) y' - y = 0$ near $x = 0$.

   (b) Attempt any **one** : **7**

   (i) Convert the equation $(x^2 - 1) y'' + (5x + 4) y' + 4y = 0$ into hypergeometric equation and solve near $x = -1$.

   (ii) Convert the equation $(x^2 - x - 6) y'' + (5 + 3x) y' + y = 0$ into hypergeometric equation and solve near $x = 3$.

3. (a) Attempt any **one** : **7**

(i) State and prove mini-max property.

(ii) Define Hermite polynomials. Give the first four Hermite polynomials.

(b) Attempt any **one** : **7**

(i) State and prove the Rodrigues' formula for the Legendre polynomials.

(ii) State and prove the orthogonality of Legendre polynomials.

4. (a) Attempt any **one** : **7**

(i) Define the Bessel function $J_p(x)$. Show that for each integer m, $J_{m+½}(x)$ is elementary.

(ii) State and prove the orthogonality kind result for $J_p(\lambda_{nx})$, where (λ_n) denotes the sequence of positive zeroes of $J_p(x)$.

(b) Attempt any **one** : **7**

(i) Show that $f(x, y) = x^2|y|$ satisfies a Lipschitz condition (in the variable y) on the rectangle $|x| \le 1$, $|y| \le 1$ but $\frac{\partial f}{\partial y}$ fails to exist at many points.

(ii) State (only) Picard's theorem. Solve the IVP : $y' = x + y$, $y(0) = 1$.

5. Attempt any **seven** : **14**

(i) Find the differential equation satisfied by the family of all concentric circles about the origin.

(ii) Prove or disprove : The set $\{1, x, x^2, x^3 \ldots\}$ is linearly independent over $\mathbb{R}$.

(iii) Find the general solution of the equation $y'' - 2y' + y = 0$.

(iv) If one particular solution of the equation $y'' + P(x)y' + Q(x)y = 0$ is given, how can we find the second independent solution ? Explain briefly.

(v) When we say that f is analytic at x_0 ? If f and g are analytic at x_0, show that f + g is also analytic at x_0.

(vi) Prove or disprove : If $f^{(n)}(0)$ exists for all $n \in \mathbb{N}$ then f is analytic at $x = 0$.

(vii) Determine the region of the convergence of the power series $\sum_{k=0}^{\infty} 2^{2k} x^{2k}$.

(viii) Find the value of $P_{2n}(0)$, where $P_{2n}(x)$ denotes the Legendre polynomial of degree 2n.

(ix) Define Lipschitz condition in the variable y. What is the importance of this condition in the solving of IVP ? Explain.
